MATHEMATICAL MODEL FOR RESONANCE IN AN ION CONVECTION PUMP

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A description has been given [1] of resonance in an ion-convection pump ICP having a pulsating voltage supply: for a certain relation between the pulsation frequency, speed of the neutral component, and length of the transfer zone (see formula (3) in [1]), there was a marked increase in the pressure difference across the stage. Here we construct a model for this.

1. Model Description. We consider the nonstationary hydraulic approximation† for the EHD equation system for a stage:

$$\frac{\partial v}{\partial x}=0,\ \rho\frac{\partial v}{\partial t}=\frac{\varepsilon\varepsilon_0}{F}\frac{\partial}{\partial x}\left[F_i\frac{E^2}{2}\right]-\frac{\partial p}{\partial x}-\xi\frac{\rho v_*^2}{2d},$$
$$\frac{\partial(F_iE)}{\partial x}=\frac{F_iq}{\varepsilon\varepsilon_0},\ F_iE=\frac{\partial(F_iU)}{\partial x},\ \frac{\partial(F_iq)}{\partial t}+\frac{\partial(F_ij)}{\partial x}=0,\ j=qv+qbE \tag{1.1}$$

subject to the boundary conditions

$$U|_{x=0}=0,\ U|_{x=l}=U_0(t); \tag{1.2}$$

$$I|_{x=0}=I_0(t),\ \left.\frac{\partial E}{\partial t}\right|_{x=0}=0; \tag{1.3}$$

$$U_0(t)=A\left|\sin\frac{\omega t}{2}\right|,\ I_0(t)=\begin{cases}k_0U_0(U_0-U^*), & U_0\geqslant U^*,\\ 0, & U_0\leqslant U_0^*.\end{cases} \tag{1.4}$$

Here A and ω are the supply-voltage pulsation amplitude and frequency, ℓ the transport-zone length, F and d the area and diameter of the channel cross section, $F_i = F_i(x, t)$ the cross section area in the space-charge zone in the stage (Fig. 1), v and p the speed and pressure averaged over the cross section for the neutral component, ρ the density, v_* is characteristic velocity, ξ is hydraulic loss coefficient, E, U, q, j, I are the field strength, potential, charge density, conduction-current density, and total current averaged over the cross section F_i, ε_0, and ε are the electrical constant and the dielectric constant, b is ionic mobility, and U* is corona striking voltage. Empirical formula (1.4) for $I_0(t)$ has been discussed in [2, 4], including the calculation of k_0. Also, (1.1) incorporates the fact that F_i = const.

The following assumptions are made.

1. The supply-voltage pulsations do not have time to influence the speed of the neutral component: $\partial v/\partial t \ll 1$, so from the first equation in (1.1) we have

$$v=\text{const}. \tag{1.5}$$

2. On account of the pulsation, the ions move along the transport zone in batches [1] with a velocity of the order of (1.5). We formalize that assumption and get q(x, t) or, which is equivalent, the electrical relaxation frequency

$$\beta(x,t)=bq/\varepsilon\varepsilon_0 \tag{1.6}$$

(see [5]), which is periodic on both arguments with periods

$$T=2\pi\omega^{-1},\ X=vT. \tag{1.7}$$

3. The F_i = x = 0 and x = ℓ are stable:

$$F_i|_{x=0}=F_0\approx 2\pi r_0,\ F_i|_{x=l}=F \tag{1.8}$$

†Equation (1.1) may be derived from the EHD equations for an element in the channel and the usual assumptions in hydraulics by analogy with [2, Chap. 1, 3, Sec. 2.2].

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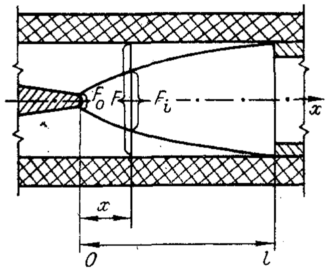

Fig. 1

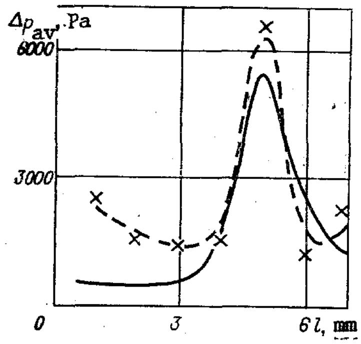

Fig. 2

(r_0 is the radius of curvature at the corona point). The second equation in (1.8) is obeyed for neutral-component speeds $v \leqslant 10$ m/sec, which are characteristic of real ICP.

4. Outside the space-charge zone, $E \ll 1$, which agrees with the quasi-one-dimensional model of (1.1) used for the stage; see pp. 72 and 142 of [3].

2. Electrical Resonance in an ICP Stage. We introduce into (1.1) the $\hat{E} = F_i E$ and similarly $\hat{U}$, $\hat{q}$, $\hat{j}$, which are the integrals of the corresponding electrical quantities over the cross section of the channel in accordance with the symbols in Sec. 1 and assumption (1.4), and we then use (1.6) to get

$$\frac{\partial \widehat{E}}{\partial x} = \frac{1}{\varepsilon\varepsilon_0}\widehat{q},\ \frac{\partial \widehat{q}}{\partial t} + \frac{\partial \widehat{j}}{\partial x} = 0,\ \widehat{E} = -\frac{\partial \widehat{U}}{\partial x},\ \widehat{j} = v\widehat{q} + \varepsilon\varepsilon_0\beta\widehat{E}. \quad (2.1)$$

By virtue of (1.2) and (1.8)

$$\widehat{U}|_{x=0} = 0,\ \widehat{U}|_{x=l} = \widehat{U}_0(t) = FU_0(t). \quad (2.2)$$

If the pulsation frequency is large enough ($\omega \gg 1$), $\hat{U}(x, t)$ approximately satisfies

$$\frac{\partial^2 \widehat{U}}{\partial t^2} - v^2\frac{\partial^2 \widehat{U}}{\partial x^2} + \bar{\beta}\frac{\partial \widehat{U}}{\partial t} - v\bar{\beta}\frac{\partial \widehat{U}}{\partial x} = \frac{v}{\varepsilon\varepsilon_0}F_0\bar{I}_0, \quad (2.3)$$

in which $\bar{\beta}$ and $\bar{I}_0$ are the means over the periods of (1.7) for (1.4) and (1.6):

$$\bar{\beta} = \frac{1}{XT}\int_0^X\int_0^T \beta(x, t)\,dx\,dt,\ \bar{I}_0 = \frac{1}{T}\int_0^T I_0(t)\,dt. \quad (2.4)$$

The first two equations in (2.1) imply

$$(\partial/\partial x)(\varepsilon\varepsilon_0\partial\widehat{E}/\partial t + \widehat{j}) = 0. \quad (2.5)$$

As the expression in parentheses is $\hat{I} = F_i I$, from (1.8), (2.5), and the first boundary condition in (1.3) we have $\varepsilon\varepsilon_0\partial\widehat{E}/\partial t + \widehat{j} = F_0 I_0(t)$, or on the basis of the last and first equations in (2.1)

$$\frac{\partial \widehat{E}}{\partial t} + v\frac{\partial \widehat{E}}{\partial x} + \beta(x, t)\widehat{E} = \frac{1}{\varepsilon\varepsilon_0}I_0(t). \quad (2.6)$$

As β and I_0 have the (1.7) periods, we apply the averaging principle for hyperbolic equations [6, 7] to (2.6) to get that for sufficiently large ω, the solutions to (2.6) are close uniformly in x and t to the solutions to the averaged equations

$$\frac{\partial \widehat{E}}{\partial t} + v\frac{\partial \widehat{E}}{\partial x} + \bar{\beta}\widehat{E} = \frac{1}{\varepsilon\varepsilon_0}F_0\bar{I}_0 \quad (2.7)$$

in which $\bar{\beta}$ and $\bar{I}_0$ are the (2.4) constants. We differentiate with respect to t and integrate with respect to x over $[0, \ell]$ successively in (2.7) and use the third equation in (2.1) and the second boundary condition in (1.3) together with the equation following from (2.1) and (2.7)

$$\frac{\partial \widehat{E}}{\partial t} = v\frac{\partial^2 \widehat{U}}{\partial t^2} - \bar{\beta}\frac{\partial \widehat{U}}{\partial x} + \frac{1}{\varepsilon\varepsilon_0}F_0\bar{I}_0,$$

to get (2.3).

We now consider the (2.2) and (2.3) boundary-value problem. The substitution $\hat{U} \to u$ in

$$\hat{U} = u \exp\left(-\frac{\bar{\beta} x}{v}\right) + \frac{x}{l} \hat{U}_0 \tag{2.8}$$

results in the standard form

$$Lu = \frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} + \bar{\beta} \frac{\partial u}{\partial t} + \frac{\bar{\beta}^2}{4} u = f(x, t), \ u|_{x=0} = u_{x=l} = 0, \tag{2.9}$$

in which

$$f(x, t) = \exp\left(\frac{\bar{\beta} x}{2v}\right)\left[-\frac{x}{l}(\hat{U}_0'' + \bar{\beta}\hat{U}_0') + \frac{v\bar{\beta}}{l}\hat{U}_0 + \frac{v}{\varepsilon\varepsilon_0} F_0 \bar{I}_0\right]. \tag{2.10}$$

The last term in the square brackets in (2.10) is small by comparison with the others on the basis of (1.4), (1.8), and (2.2) as r_0 and k_0 are small [2, 4], and as (2.9) contains the friction $\bar{\beta}\partial u/\partial t$, all the solutions to the homogeneous boundary-value problem $Lu = 0$, $u|_{x=0} = 0$, $u|_{x=\ell}$ decrease exponentially for $t \to +\infty$, and it is thus sufficient to calculate the steady-state solution to (2.9).

We discard the small term $(v/\varepsilon\varepsilon_0)F_0\bar{I}_0$ in (2.10) and replace $\hat{U}_0(t)$ by a standard harmonic, after which we represent $f(x, t)$ and the solution to (2.9) for a specified t as series in sines in $[0, \ell]$, and then simple calculations analogous with those of [8, p. 136] give

$$u(x, t) = \sum_{n=1}^{\infty}\left(a_n \cos\omega t + b_n \sin\omega t + \frac{r_n}{\omega_n^2}\right)\sin\frac{\pi n x}{l}, \tag{2.11}$$

in which

$$\omega_n = \sqrt{\left(\frac{n v \pi}{l}\right)^2 + \frac{\bar{\beta}^2}{4}} \quad (n = 1, 2, \ldots); \tag{2.12}$$

$$a_n = \frac{c_n(\omega_n^2 - \omega^2) - d_n\bar{\beta}\omega}{(\omega_n^2 - \omega^2)^2 + \bar{\beta}^2\omega^2}, \quad b_n = \frac{c_n\bar{\beta}\omega + d_n(\omega_n^2 - \omega^2)}{(\omega_n^2 - \omega^2)^2 + \bar{\beta}^2\omega^2}; \tag{2.13}$$

$$\begin{gathered} c_n = -\frac{2r_n}{3} + \frac{\omega}{\bar{\beta}} d_n, \quad r_n = \frac{4AFv\bar{\beta}}{l}\frac{1 - e^{\lambda}(-1)^n}{\lambda^2 + \pi^2 n^2}, \\ d_n = \frac{8AF\bar{\beta}\omega_n}{3}\left[\frac{e^{\lambda}(-1)^n}{\lambda^2 + \pi^2 n^2} + \frac{\lambda[1 - e^{\lambda}(-1)^n]}{(\lambda^2 + \pi^2 n^2)^2}\right], \quad \lambda = \frac{\bar{\beta} l}{2v}. \end{gathered} \tag{2.14}$$

The (2.11) series converges absolutely and uniformly in $x \in [0, l]$, $t \in (-\infty, +\infty)$. We see from (2.13) that the amplitude of harmonic n in (2.11) is

$$A_n = \sqrt{a_n^2 + b_n^2} = \frac{\sqrt{c_n^2 + d_n^2}}{\sqrt{(\omega_n^2 - \omega^2)^2 + \bar{\beta}^2\omega^2}}$$

so there is a maximum at $\omega_n = \omega$. Harmonic n in (2.11) thus resonates at the (2.12) frequency n.

We consider the principal resonance $\omega_1 = \omega$:

$$\sqrt{\left(\frac{v\pi}{l}\right)^2 + \frac{\bar{\beta}^2}{4}} = \omega. \tag{2.15}$$

In the particular case $\bar{\beta}/\omega \ll 1$, we have $v\pi/\ell = \omega$, which as $\omega = 2\pi f$ coincides with the resonance relation (3) of [1] found by experiment, so (2.15) refines the observed resonance formula of [1] provided that the electrical relaxation frequency of (1.6) is comparable with the supply-voltage pulsation frequency, and then (2.12) gives the higher resonant frequencies.

3. Calculating Δp_{av} and Comparison with Experiment. We integrate the second equation in (1.1) over $[0, \ell]$ and use (1.5) and (1.8) to get

$$\Delta p = \frac{\varepsilon\varepsilon_0}{2F}[FE^2(l, t) - F_0E^2(0, t)] - \frac{\xi\rho v^2 l}{2d}$$

$(\Delta p = p(l, t) - p(0, t))$. As r_0 is small, $F_0 \ll 1$ in (1.8) and we can put approximately $\Delta p = (\varepsilon\varepsilon_0/2)E^2(l, t) - \xi\rho v^2 l/2d$, so

$$\Delta p_{av} = (\varepsilon\varepsilon_0/2)(E^2)_{av} - \xi\rho v^2 l/2d \quad (3.1)$$

(Δp_{av} and $(E^2)_{av}$ are the average values of Δp and $E^2(\ell, t)$ over the period [0, T]).

Simple calculations on the basis that $E(\ell, t) = F^{-1}\hat{E}(\ell, t)$ with the third formula in (2.1) and with (2.11) and (3.1) give

$$\Delta p_{av}(l) = (\varepsilon\varepsilon_0/2)D(l) - \xi\rho v^2 l/2d, \quad (3.2)$$

in which

$$D = \frac{44A^2}{9\pi^2 l^2} + \frac{4A\pi}{3Fl}(A - 3R)e^{-\lambda} + \frac{\pi^2}{F^2 l^2}\left(\frac{A^2 + B^2}{2} + R^2\right)e^{-2\lambda},$$
$$A = \sum_{n=1}^{\infty}(-1)^{n+1}a_n,\ B = \sum_{n=1}^{\infty}(-1)^{n+1}b_n,\ R = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{r_n}{\omega_n^2}, \quad (3.3)$$

with a_n, b_n, r_n, ω_n, λ derived from (2.12)-(2.14). The (3.3) series converge absolutely.

Figure 2 compares calculations on $\Delta p_{av}(\ell)$ from (3.2) with experiment for an input pulsation frequency f = 100 Hz for an organosilicon liquid having $\rho = 850$ kg/m³ and $\varepsilon = 2.4$ (solid line from theory, dashed line from experiment). We assume $v \approx 1$ m/sec (from experiment) and $\xi \ll 1$ (zero flow), with n = 10 in (3.3). The two curves have been constructed for $\bar{\sigma} \approx 5 \cdot 10^{-9}$ 1/Ω·m, so from (1.6) we have $\beta = \bar{\sigma}/\varepsilon\varepsilon_0 \approx 200$ Hz.

The qualitative theoretical resonance pattern coincides with the observed one. The relative error in (3.1) in the resonant region in these cases is not more than 20%.

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